Birkhoff normal forms for betatronic motion and stability indicators

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Celestial and beam dynamics models

Birkhoff normal forms

Dyamic aperture

Lyapunov and noise induced errors

Conclusions

CELESTIAL MECHANICS

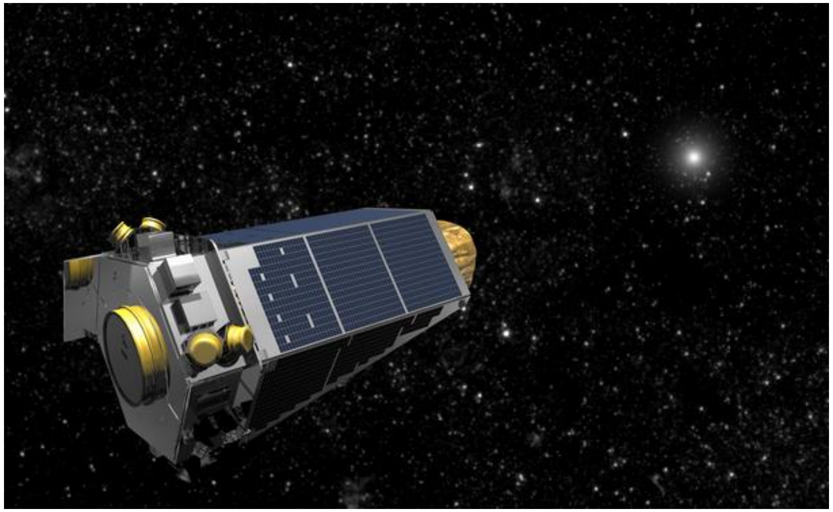
MODELS

Fronteers in Physics

Ultimate structure of matter:
the Higgs boson and beyond

Large Hadron Collider

Planetary systems in our galaxy, earth like worlds, life beyond the earth



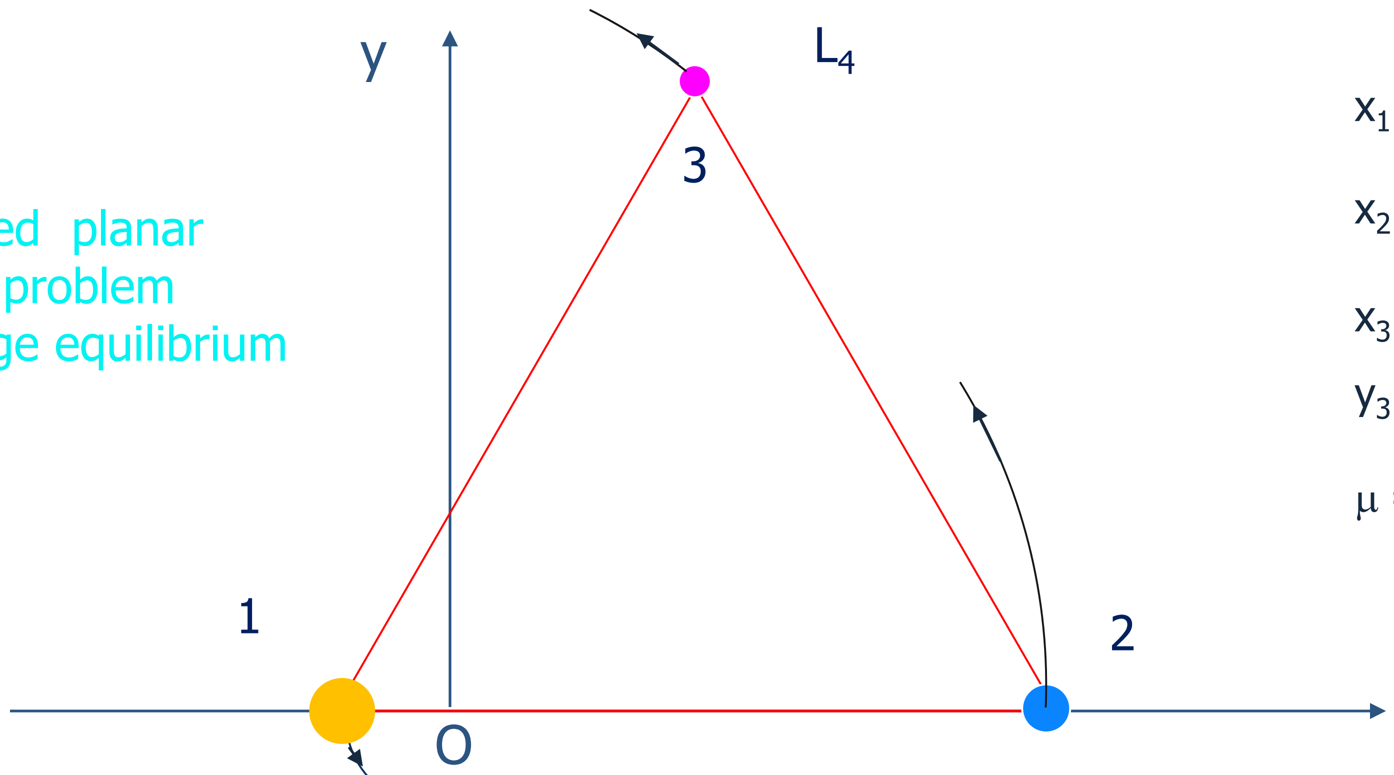
Kepler Space Telescope

Models in celestial mechanics

A planet in the solar system and of a proton in a ring have dynamical analogies and comparable stability times.

The 3 body problem: sun, Jupiter with $m_1 > m_2$ on circular orbits and a satellite with $m_3 \to 0$. Equilibrium at vertices of an equilateral triangle.

restricted planar
3 body problem
Lagrange equilibrium



$x_1 = -\mu$

$x_2 = 1-\mu$

$x_3 = \frac{1}{2} - \mu$

$y_3 = \sqrt{3}/2$

$\mu = m_2/(m_1+m_2)$

Birkhoff normal forms for betatronic motion

Scaling coordinates and time ($r_{12}=1$, $T=2\pi$) the Hamiltonian in corotating system where V is the gravitational potential $v_x=p_x+y$ $v_y=p_y-x$.

$$H= \tfrac{1}{2} (p_x^2+p_y^2) + yp_x-xp_y + V(x,y)$$

Normal form near L_4 where $X=Y=0$ and $J_x= \tfrac{1}{2}(X^2+P_x^2)$ $J_y = \tfrac{1}{2}(Y^2+P_y^2)$

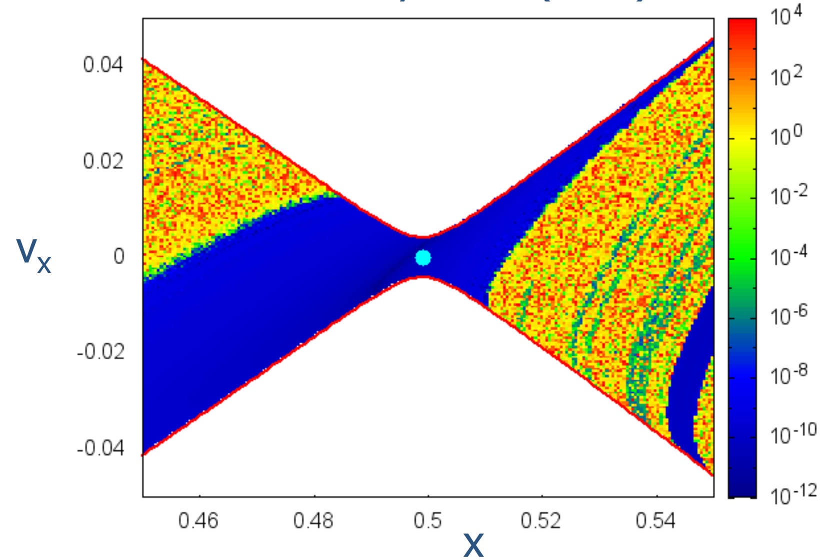
$$H= \omega_1 J_x + \omega_2 J_y + H_3 (J_x, J_y) + \dots + R_N \qquad \omega_1 \omega_2 <0$$

H has a saddle at L_4 no Lyapunov stability !!

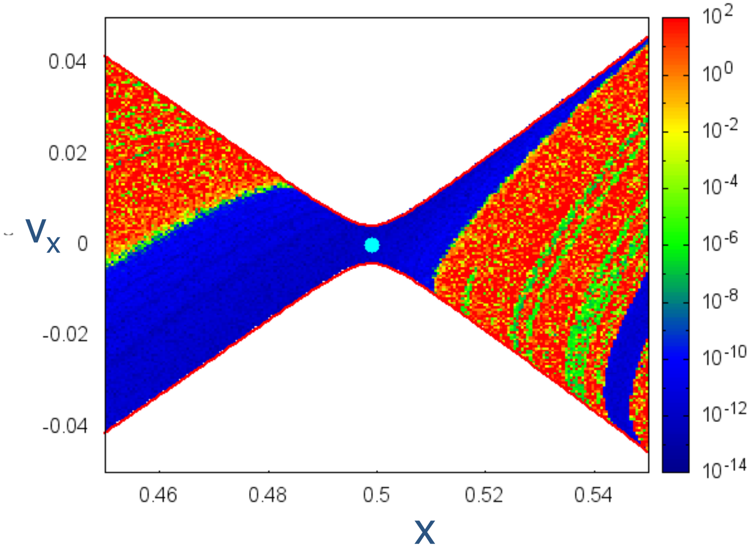
Birkhoff normal forms for betatronic motion

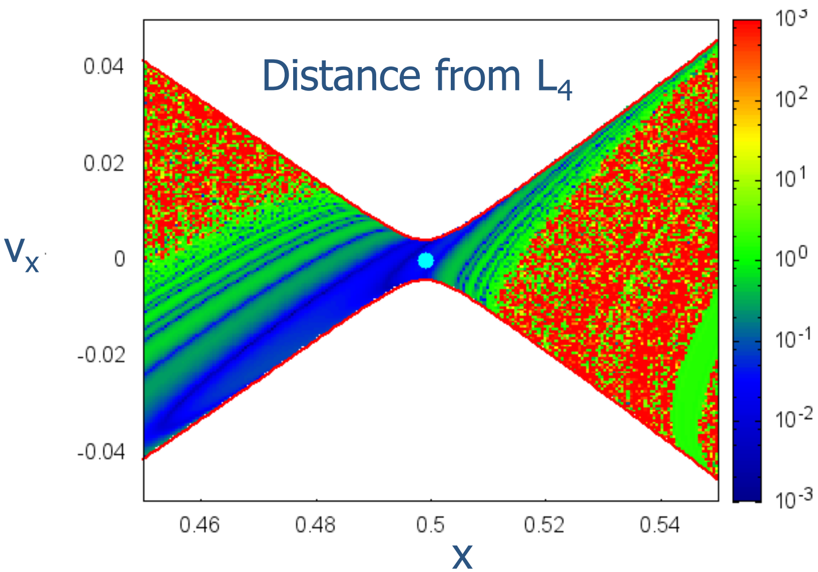
Error plots. Poincaré section $y=y_c$ $v_y>0$ $H=H_c+10^{-5}$ $H_c \sim 1.5$ $t_{max}= 200$ T

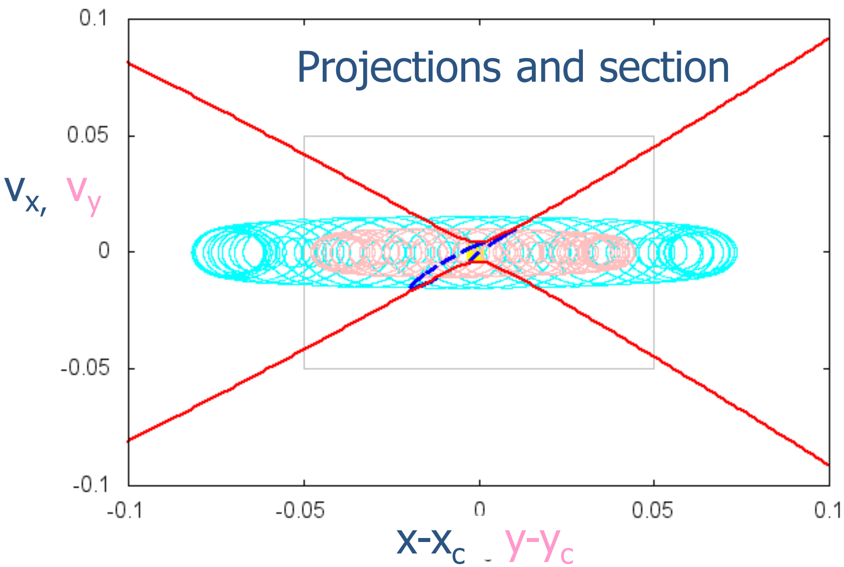
Reversibility error (REM)



Lyapunov error (LE)







The Hénon-Heiles model.

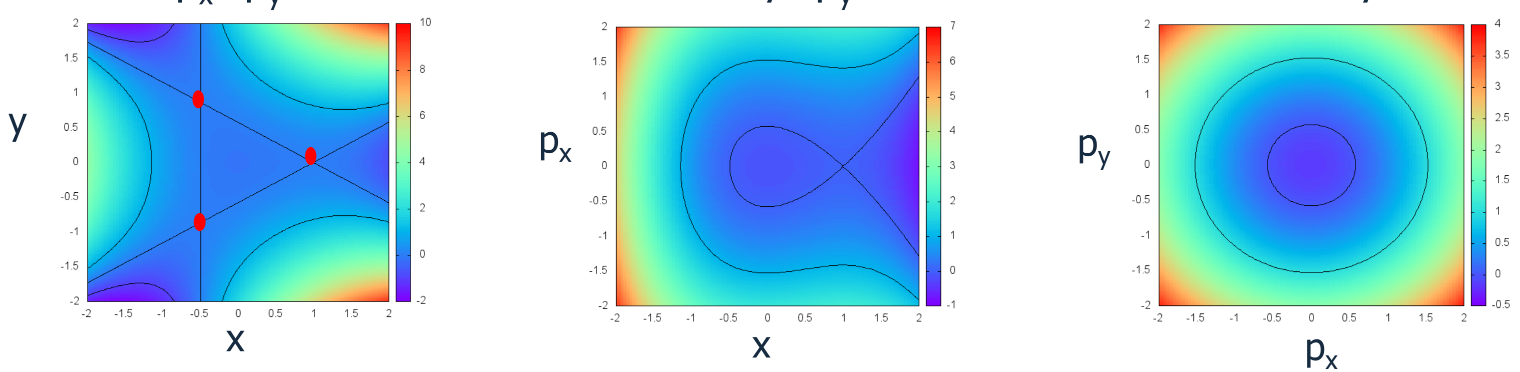
Motion of a star in an elliptical galaxy

$$H = T + V = \tfrac{1}{2}\,(p_x^2+p_y^2) + \tfrac{1}{2}\,(x^2+y^2) - x^3/3 + xy^2$$

V=0 has a minimum at x=y=0 and three saddle points at the vertices of an equilateral triangle where V=1/6

stability H<1/6 boundary H=1/6

$p_x=p_y=0$ $y=p_y=0$ $x=y=0$



Isolines of H : innerst curve H=1/6 stability boundary

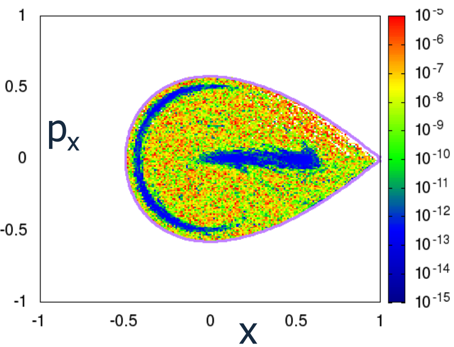
Birkhoff normal forms and betatronic motion

Error plots Poincaré section $y=0$ $p_y>0$ $H=E$, boundary $H(x,p_x,0,0)=E$

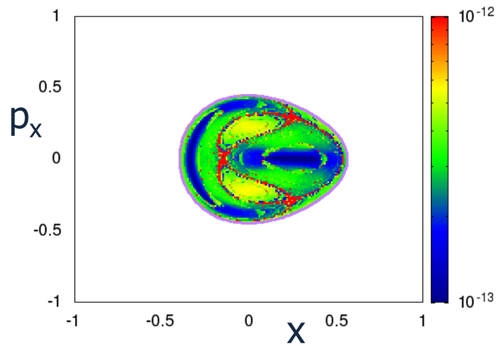
Plots in x,p_x plane for y_0, p_{y0} fixed, boundary $H(x,p_x,y_0,p_{y0})=1/6$ $t_{max}=20T$

Reversibility error (REM) P. section y=0

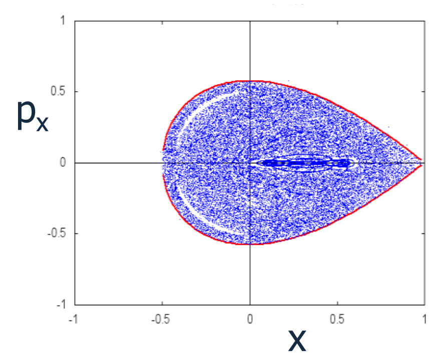
Orbits P section y=0



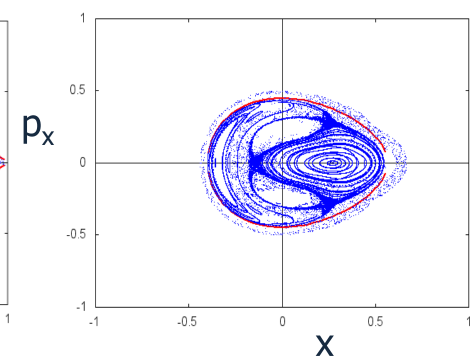
E=1/6



E=0.1



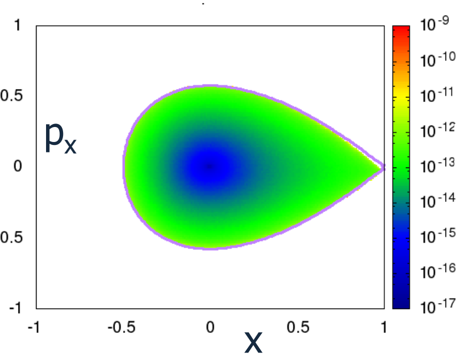
E=1/6



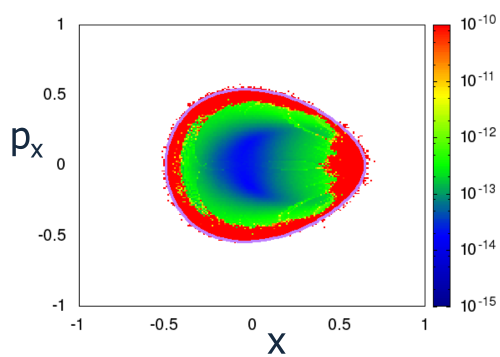
E=0.1

REM x p_x plane

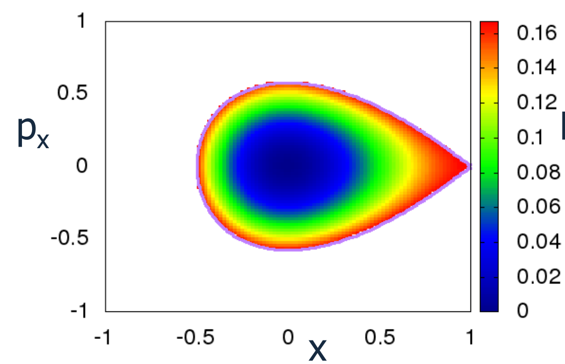
H value x, p_x plane



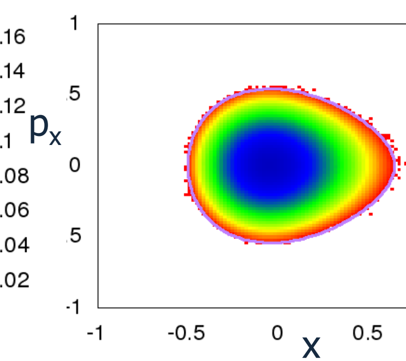
$y_0=p_{y0}=0$



$y_0=p_{y0}=0.2$



$y_0=p_{y0}=0$



$y_0=p_{y0}=0.2$

Tools for dynamic analysis

- Normal forms. Hamiltonian invariant under a symmetry group up to a remainder. Basically an analytic tool.

- Frequency map error. The FFT of a quasi periodic signal $n=2^m$ gives tunes $\nu=(\nu_1, \nu_2)$. Error $e(n) = \| \nu(n) - \nu(n/2) \|$

- Lyapunov error . Induced by an initial displacement

- Reversibility error. Induced by noise or round-off

BEAM DYNAMICS

MODELS

Beam dynamics models: betatronic motion

Unlikely the Kepler problem the circular motion of a charge under a uniform magnetic field **B** is not stable (drift along **B**).

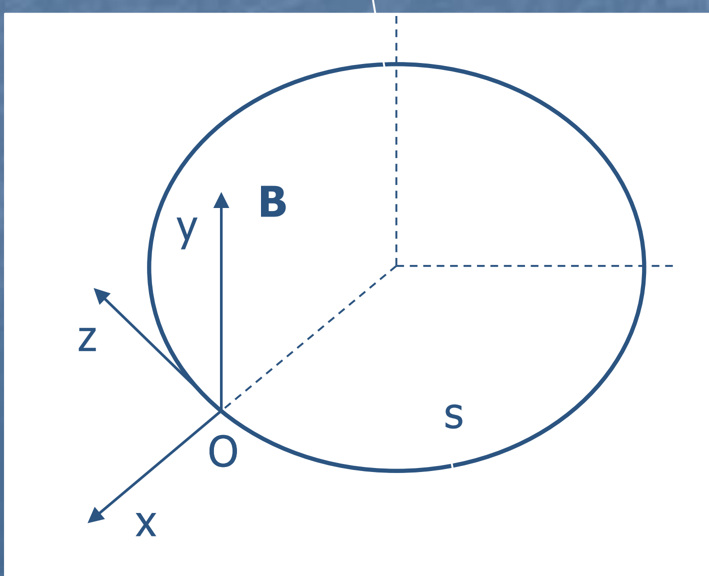
$$\boxed{H = \tfrac{1}{2}(p_x^2+p_y^2) + \tfrac{1}{2}\,\frac{x^2}{R^2} + V(x,y,s)}$$

$s=v_0t$ $\qquad$ $p_x=dx/ds$

Multipoles contribution

$$V = -\,1/2\;K_1(s)\,(x^2-y^2) - 1/6\;K_2(s)\,(x^3-3xy^2) + \dots$$

Explicit map for thin multipoles K_m with $m \geq 3$



Birkhoff normal forms and betatronic motion

Linear lattice $M(x)= L\, x$ conjugated to a rotation

$$L= W\ R(\omega)\ W^{-1} \qquad W = \begin{pmatrix} \beta^{1/2} & 0 \\ -\alpha\, \beta^{-1/2} & \beta^{-1/2} \end{pmatrix} \qquad \begin{pmatrix} x' \\ p'_x \end{pmatrix} = W_x^{-1} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

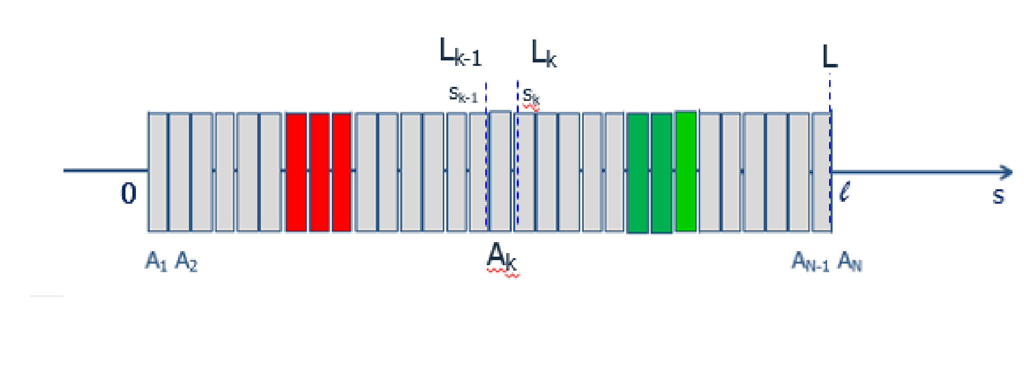
normal form

Courant-Sneider coordinates x', p'_x Change of section from s_{k-1} to s_k

$$L_k = A_k\, L_{k-1}\, A_k^{-1} \qquad L_k=L(s_k)$$

Exact recurrence for $\beta_k=\beta(s_k)$,

$\alpha_k=\alpha(s_k)$ and phase advance



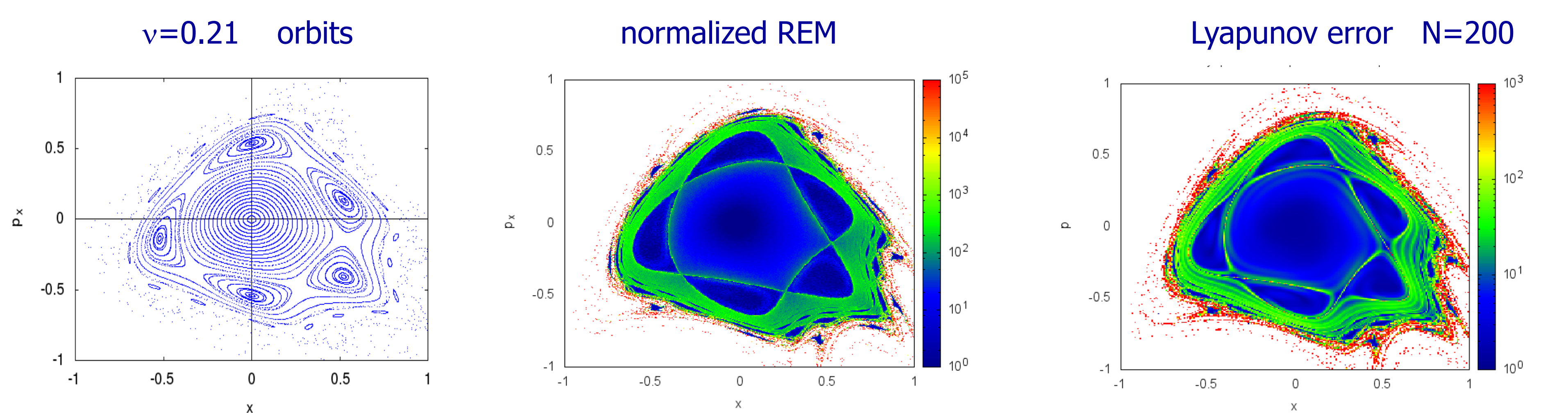
Birkhoff normal forms and betatronic motion

The 2D Hénon map

One turn map for linear lattice with a thin sextupole in scaled coordinates

$\mathbf{X} = \frac{1}{2}\, \beta_x^{3/2}\, k_2\, W^{-1}\, \mathbf{x}$ for a flat beam

$$\begin{pmatrix} X_{n+1} \\ P_{n+1} \end{pmatrix} = R(\omega) \begin{pmatrix} X_n \\ P_n + X_n^2 \end{pmatrix}$$

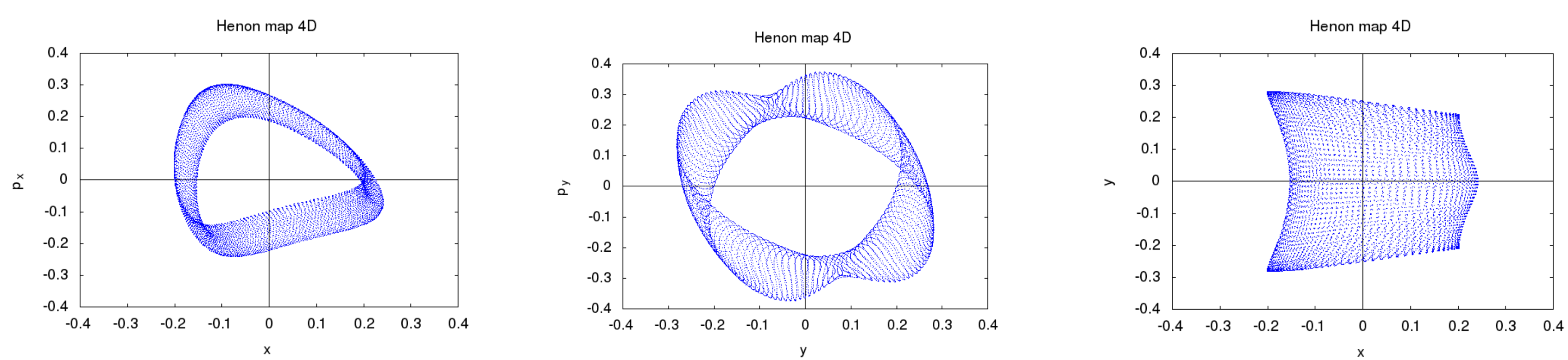


Birkhoff normal forms and betatronic motion

The 4D Hénon map

One turn map for a linear lattice with a thin sextupole in scaled coordinates

$$\begin{pmatrix} X_{n+1} \\ P_{x\,n+1} \\ Y_{n+1} \\ P_{y\,n+1} \end{pmatrix} = \begin{pmatrix} R(\omega_x) & 0 \\ 0 & R(\omega_y) \end{pmatrix} \begin{pmatrix} X_n \\ P_{x\,n} + X_n^2 - \beta Y_n^2 \\ Y_n \\ P_{y\,n} - 2\beta X_n Y_n \end{pmatrix} \qquad \beta = \frac{\beta_y}{\beta_x}$$



$\nu_x = (3 - \sqrt{5})/2$ $\nu_y = \sqrt{2} - 1$ $\beta = 1$ projection of orbit with $x_0 = y_0 = 0.2$ $p_{x0} = p_{y0} = 0$

BIRKHOFF NORMAL FORMS

Birkhoff normal forms and betatronic motion

The one turn (superperiod) map M for thin multipoles is a polynominal which can be truncated to order N. In Courant-Sneider coordinates

$$M_N(\mathbf{x}) = R(\omega)\,(\mathbf{x} + P_2(\mathbf{x}) + .. + P_N(\mathbf{x})\,) \qquad M(\mathbf{x}) = M_N(\mathbf{x}) + O(|\mathbf{x}|^{N+1})$$

A nonlinear symplectic tranformation $\mathbf{x} = \Phi(\mathbf{X})$ changes $M(\mathbf{x})$ into a new map $U(\mathbf{X})$ which is invariant under the group generated by $R(\omega)$

$$\boxed{M(\mathbf{x}) = \Phi \circ U \circ \Phi^{-1}(\mathbf{x})} \qquad U(R\mathbf{X}) = RU(\mathbf{X})$$

$$\boxed{U(\mathbf{X}) = R\,\exp(D_H)\,\mathbf{X}} \qquad \boxed{H(R\mathbf{X}) = H(\mathbf{X})}$$

$$\boxed{\Phi(\mathbf{X}) = \exp(\,D_G\,)\,\mathbf{X}}$$

H is the interpolating Hamiltonian, Φ a symplectic coordinates change

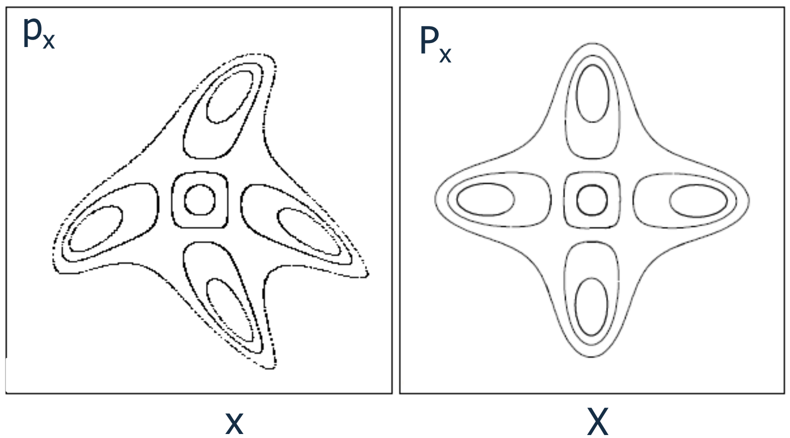
Birkhoff normal forms and betatronic motion

If ω is non resonant ($\nu=\omega/2\pi$ irrational) $R(\omega)$ generates a continuous group of rotations $J=\frac{1}{2}(P_x{}^2+X^2)$ is the invariant.

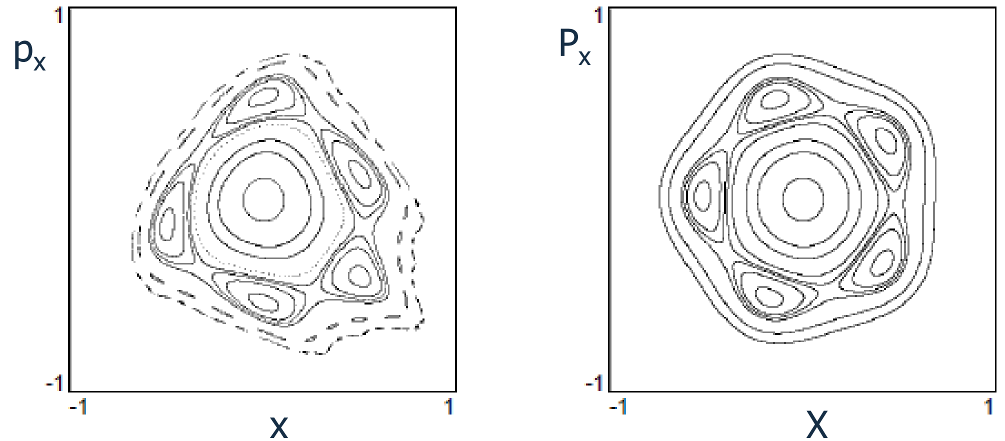
If $\omega=\omega_R$ is resonant or quasi resonant $\omega=\omega_R+\varepsilon$ ($\nu_R=m/q$ $\varepsilon<<\omega_R$) $R(\omega_R)$ generates a discrete group of rotations and

$$\boxed{H(J,\theta) = \sum_k h_k(J)\cos(k\,q\,\theta + \alpha_k)}$$

is invariant under the group. The level lines of H exhibit a chain of q islands



Quasi resonant normal forms for v=0.255 (q=4) in x,p_x and X,P_X planes



Orbits for v=0.21 in x,p_x plane and quasi resonanant normal form (q=5) in the X,P_X plane

Nekhoroshev stability estimates and analyticity

The series defining the normal forms are divergent due to singularities associated to resonances. Conjugation with normal form up to a remainder

$$M = \Phi_N \circ (U_N + E_N) \circ \Phi_N^{-1} \qquad U_N = R \exp(D_{H_N})$$

For a 2D where $r = (X^2 + P_x^2)^{1/2} = (2J)^{1/2}$ estimate

$$\boxed{|E_{N-1}| < A\ (r/r_N)^N} \qquad r < r_N = 1/(CN)$$

Minimum achieved for $N = N_* = (e\, C\, r)^{-1}$ and $r/r_{N_*} = e^{-1}$. In a disc of radius r

$$|E_{N_*}| < A \exp(-N_*) = \exp(-r_*/r) \qquad r_* = (eC)^{-1}$$

Orbits starting in a disc $r/2$ remain in disc of radius r for $n|E_{N*}| < r/2$

$$\boxed{n < \tfrac{1}{2} A r \exp(r_*/r)}$$

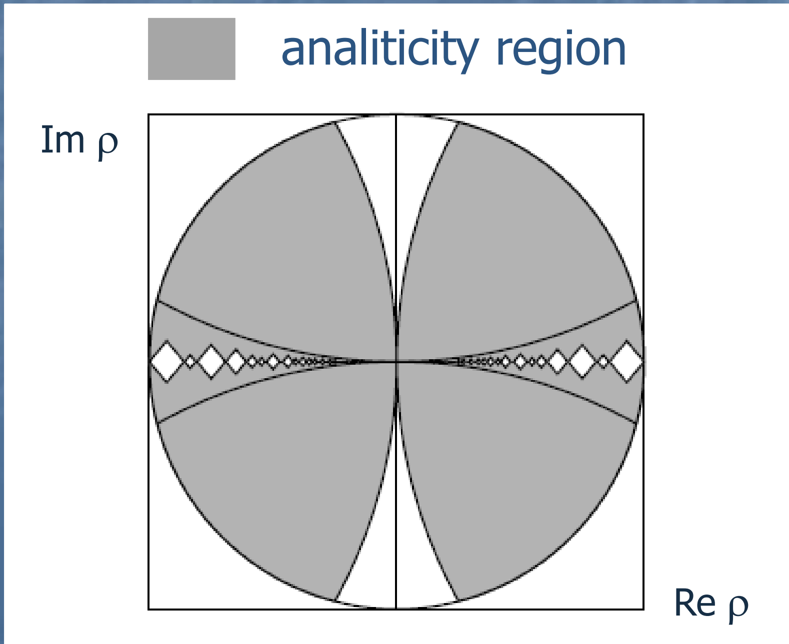
Singularities of normalizing transformations

The Birkhoff series diverge due to an accumulation at the origin of the complex $\rho=2J$ plane of singularities associated to the resonances. If $\omega=2\pi\, p/q+\varepsilon$ for $\varepsilon \to 0$ the conjugation function Φ behaves as a geometric series. Φ has a pole at $\rho=\rho_q$ where $\Omega= dH/dJ$ is resonant.

$$\Omega=\omega+\rho\,\Omega_2 = 2\pi\, p/q \qquad \rightarrow \qquad \rho_q = -\varepsilon/\Omega_2$$

In the generic case varying ρ the frequency crosses infinitely many resonances. The leading ones correspond to the continued fraction expansion p_j/q_j of the tune ν and are located approximately at $\rho_j=-\varepsilon_j/\Omega_2$ where $\varepsilon_j = \omega - 2\pi\, p_j / q_j$

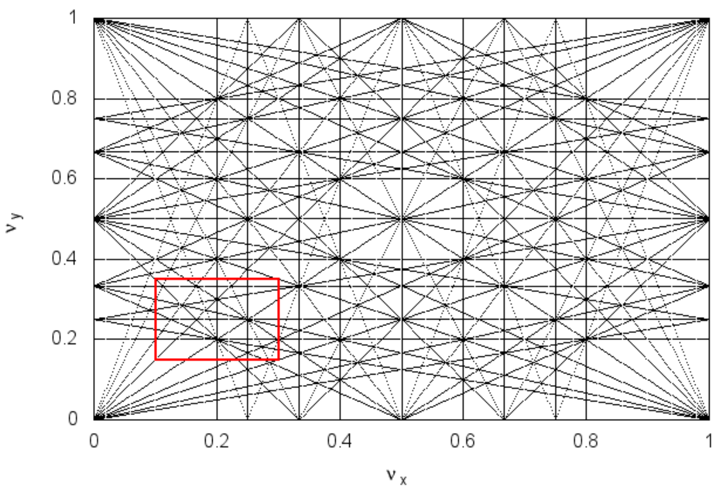
A rigorous analysis confirms this picture. If $\rho_j>0$ we have a true resonance (chain of islands). The resonance is virtual if $\rho_j<0$



DYNAMIC APERTURE

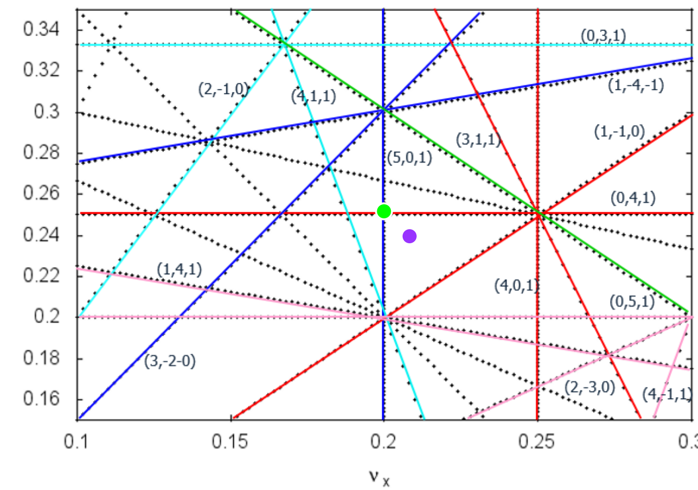
Birkhoff normal forms and betatronic motion

Work point. From $H = \omega_x J_x + \omega_y J_y + \frac{1}{2}(h_{11} J_x^2 + 2h_{12} J_x J_y + h_{22} J_y^2)$ to any resonance $k_x\Omega_x + k_y\Omega_y = 2\pi\, m$ corresponds a line $a_x J_x + a_y J_y = b$ in action space



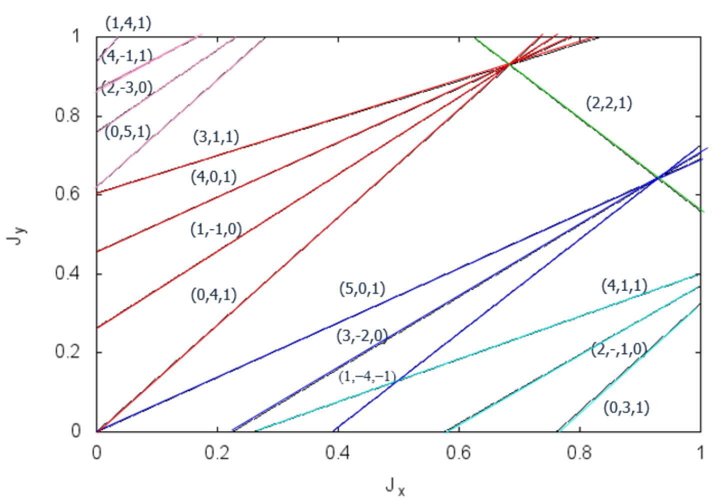
Tune plot

$|k_x| + |k_y| \le 5$



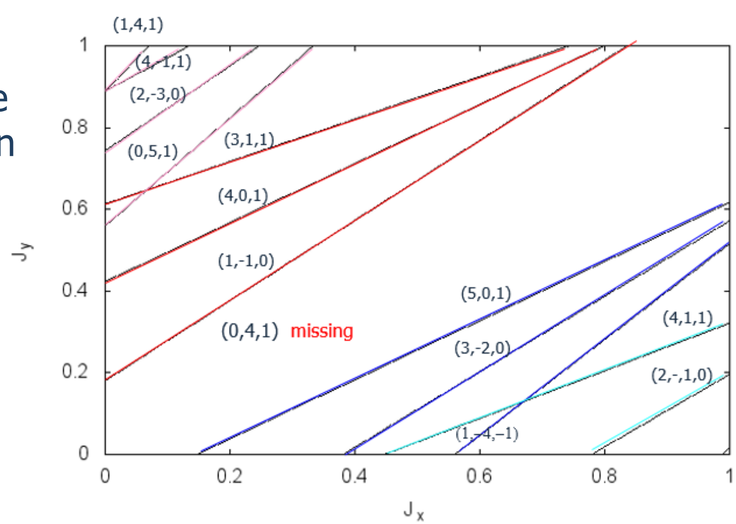
Tune plot

magnif.



Resonance lines action space

$\nu_x = 1/5$
$\nu_y = 1/4$



Resonance lines action space

$\nu_x = 0.21$
$\nu_y = 0.19$

Birkhoff normal forms and betatronic motion

Short term dynamic aperture of 2D Hénon map

Boundary of stability domain of H for unstable resonances.

Resonance 0: as $\omega \to 0$ interpolating Hamiltonian $H = \frac{1}{2}\,\omega\,(P^2+X^2) - X^3/3$

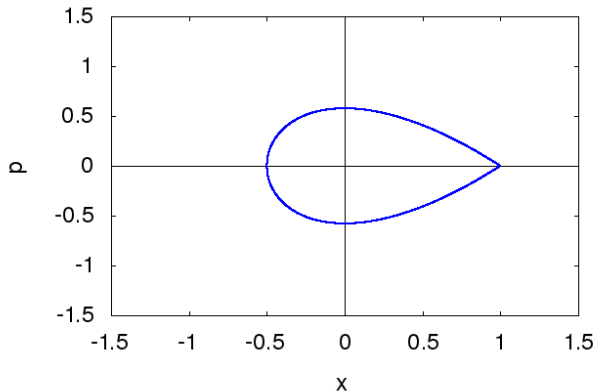
Scaling $X=\omega x$, $P= \omega p$ and $H=\omega^3 h$ boundary $h=1/6$

$$\boxed{h - \frac{1}{6} = \frac{p^2}{2} - \frac{1}{6}(1-x)^2\,(1+2x) = 0}$$

Resonance 1/3: as $\varepsilon=\omega-2\pi/3 \to 0$ interpolating Hamiltonian

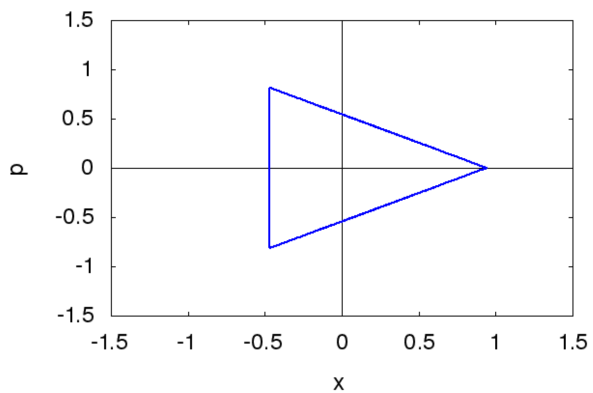
$$\boxed{H= \varepsilon\, J- 4\, J^{3/2} \cos(3\theta)}$$

$\nu \to 0$



Dynamic aperture of Henon map from resonant normal forms

$\nu - 1/3 \to 0$



Birkhoff normal forms and betatronic motion

At the critical saddle points of $h = 4/27$ After a scaling with ε

$$h - \frac{4}{27} = \frac{1}{2}\left(1+\frac{3x}{\sqrt{2}}\right)\left(p^2-\frac{1}{3}\left(x-\frac{2\sqrt{2}}{3}\right)^2\right) \le 0$$

Short term dynamic aperture of 4D Hénon map

For $\omega_1=\omega_2=\omega \rightarrow 0$ after scaling $X= \omega\, x, \ldots, P_y=\omega\, p_y$ and $H= \omega^3 h$ where h is Hénon-Heiles hamiltoian

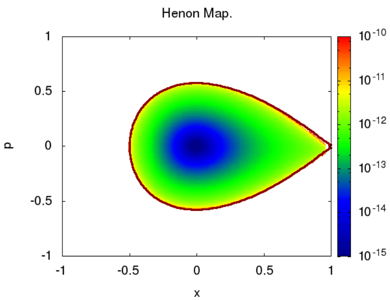
$$\boxed{h - \frac{1}{6} = \frac{p_x^2+p_y^2}{2} + \frac{1}{6}(1+2x)\left(3y^2-(1-x)^2\right) = 0}$$

stability region $h<1/6$ boundary $h=1/6$

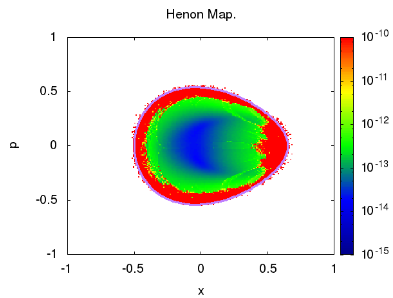
Birkhoff normal forms and betatronic motion

Dynamic aperture and REM error plots for $\nu_x = \nu_y = 0.01$ Iterations n=5000

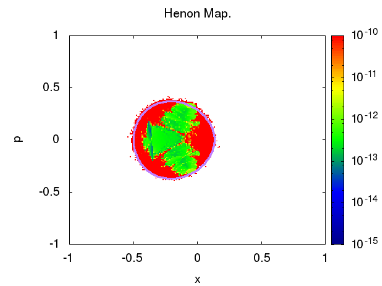
Plots in x, p_x y, p_y and x, y planes. Coordinates scaled by $\omega = 2\pi\nu_x$



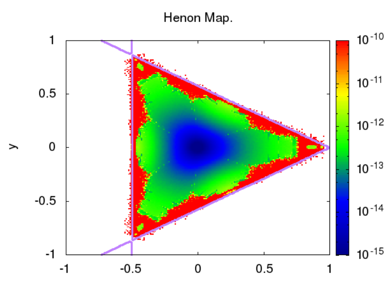
$p_y=0$ $y=0$



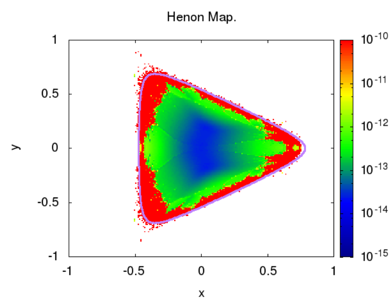
$y=0.2$



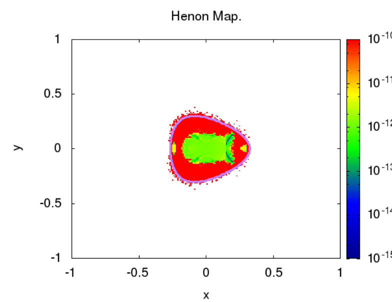
$y=0.5$



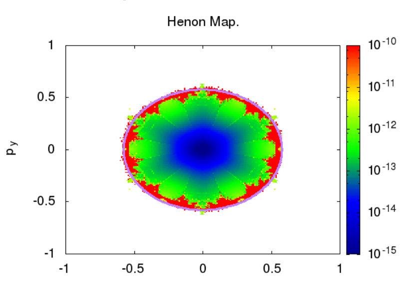
$p_y=0$ $p_x=0$



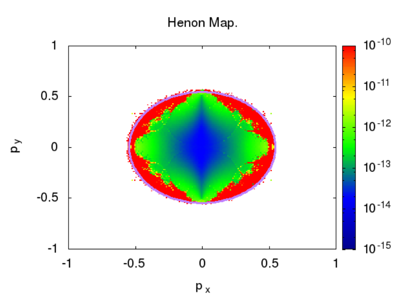
$p_x=0.2$



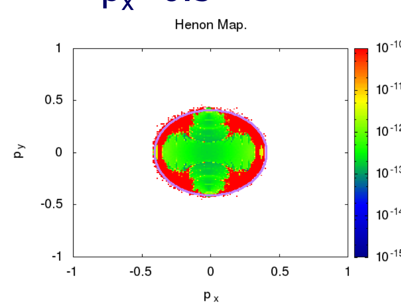
$p_x=0.5$



$p_x=0$ $x=0$



$x=0.2$



$x=0.5$

x p_x plane

purple curve analytic boundary

x y plane

Iterations 5000

y p_y plane

coordinates scaled by ω

LYAPUNOV AND NOISE ERRORS

Birkhoff normal forms and betatronic motion

Errors due do a small displacement or small noise allow stability assessments

Lyapunov error LE

Small initial displacement: let iterates of $\mathbf{x}_0$ and $\mathbf{x}_0 + \varepsilon\,\eta$ $\|\eta\|=1$ be $\mathbf{x}_n$ and $\mathbf{x}_n + \varepsilon\,\eta_n + O(\varepsilon^2)$ Normalized Lyapunov error

$$\boxed{e_L(n,\eta) = \|\eta_n\|} \qquad \eta_n = DM(x_{n-1})\,\eta_{n-1}$$

or

$$e_L(n,\eta) = \|A_n\,\eta\| \qquad A_n = DM^n(x_0)$$

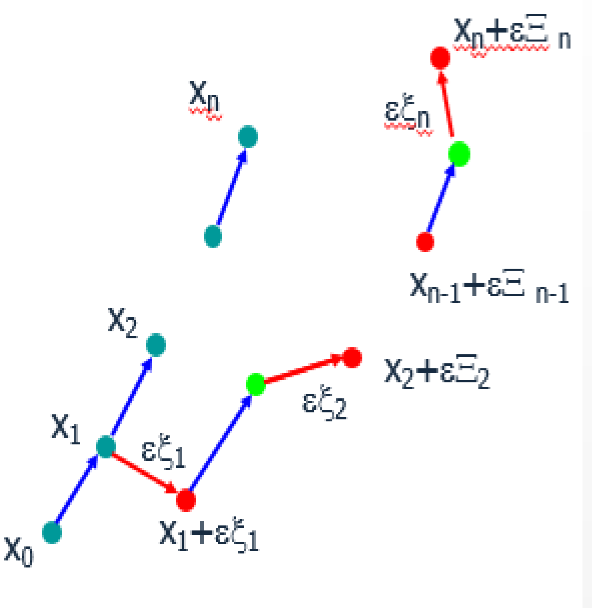
DM denotes the tangent map. To have a result independent from the vector η we sum over the errors for any orthonormal basis obtaining

$$\boxed{e_L(n) = (\,\mathrm{Tr}(A_n^T A_n)\,)^{1/2}}$$

Forward error FE

Noise of vanishingly small amplitude: ξ_n a random vector $<\xi_n\, \xi_m^T> = I\, \delta_{nm}$

$$\mathbf{x}_{\varepsilon,n} = M(\mathbf{x}_{\varepsilon,n-1}) + \varepsilon\, \xi_n = \mathbf{x}_n + \varepsilon\, \Xi_n + O(\varepsilon^2)$$



Global stochastic perturbation

$$\Xi_n = \lim_{\varepsilon \to 0} \frac{\mathbf{x}_{\varepsilon,n} - \mathbf{x}_0}{\varepsilon}$$

non homogeneous recurrence

$$\boxed{\Xi_n = DM(\mathbf{x}_{n-1})\ \Xi_{n-1} + \xi_n \qquad \Xi_0 = 0}$$

The normalized forward error is defined by

$$\boxed{e_F(n) = <\Xi_\mathbf{n} \cdot \Xi_\mathbf{n}>^{1/2}} = \left(\sum_{k=0}^{n-1} \mathrm{Tr}(\,B_k(n)\, B^T_k(n)\,) \right)^{1/2} \qquad B_k(n) = DM^{\,k}(\mathbf{x}_{n-k})$$

Notice $e^2_F(n)$ is the trace of covariance matrix $<\ \Xi_\mathbf{n}\ \Xi_\mathbf{n}^{\ T}>$

Reversibility error RE

with respect to the initial condition after n iterations forward and backwards with noise. Let $\mathbf{x}_{\varepsilon,-m,n}$ be the error after n iterations with M and m with M^{-1}

$$\mathbf{x}_{\varepsilon,-m,n} = M^{-1}(\mathbf{x}_{\varepsilon,-m+1,n}) + \varepsilon\,\xi_{-m} = \mathbf{x}_{n-m} + \varepsilon\,\Xi_{-m,n} + O(\varepsilon^2)$$

Global stochastic process

$$\boxed{\Xi_n^R = \lim_{\varepsilon\to 0} \frac{\mathbf{x}_{\varepsilon,-n,n} - \mathbf{x}_n}{\varepsilon} = \Xi_n + \sum_{k=0}^{n-1} DM^{-k}(\mathbf{x}_k)\,\xi_{-(n-k)}}$$

Error is defined by

$$\boxed{e_R(n) = \langle \Xi_n^R . \Xi_n^R \rangle^{1/2}} = \left(\sum_{k=0}^{n-1} \mathrm{Tr}\,(C_k(n) C_k^T(n)) + \sum_{k=0}^{n-1} \mathrm{Tr}\,(AI_k(n)\,AI_k^T(n)) \right)^{1/2}$$

where $AI_k = DM^{-k}(\mathbf{x}_k)$ and $C_k = AI_n B_k$

Linear maps

Errors asymptotics elementary for linear maps $DM(\mathbf{x}) = L\,x$

$$e_L(n) = \left(\mathrm{Tr}\left((L^n)^T L^n \right) \right)^{1/2} \qquad e_F(n) = \left(\sum_{k=0}^{n-1} e_L^2(k) \right)^{1/2}$$

- elliptic fixed point

  $e(n) \sim 1$ for LE $\qquad e(n) \sim n^{1/2}$ for FE, RE

- parabolic f. p.

  $e(n) \sim n$ for LE $\qquad e(n) \sim n^{3/2}$ for FE, RE

- hyperbolic f. p.

  $e(n) \sim e^{\lambda n}$ for LE, FE, RE

Power law growth and oscillations

Rotation:

$L=R(\omega)$ $e_L(n)=\sqrt{2}$ $e_F(n) =(2n)^{1/2}$

$L = \Phi\, R(\omega)\, \Phi^{-1}$ $e_L(n)= \left(A - (A-2) \cos(2n\omega) \right)^{1/2}$ $A \geq 2$ $e_L(n)$ oscillates

F

Integrable map in normal form $M(\mathbf{x})= R(\Omega(||\mathbf{x}||^2/2)\, \mathbf{x}$

$$\boxed{e_L^2(n) = 2 + (\Omega' \; ||\mathbf{x}||^2)^2 \; n^2}$$

$e_F(n) \sim n^{3/2}$ $e_R(n)= e_F(2n)$

Integrable map not in normal form $M= \Phi\, R(\Omega)\, \Phi^{-1}$ power law error growth with oscillations.

Birkhoff normal forms and betatronic motion

Averaging on oscillations

Local error growth rate given by

$$\boxed{De(n) = \frac{d \log e(n)}{d \log n} \qquad n \in \mathbb{R}}$$

$$De(n) = \frac{\log e(n+1) - \log e(n)}{\log (n+1) - \log n} \qquad n \in \mathbb{N}$$

$$D\, n^{\alpha} = \alpha \qquad\qquad D\; e^{\lambda n} = \lambda\; n \qquad\qquad n \in \mathbb{R}.$$

To damp oscillations of De(n) double average was proposed

$$\boxed{Y(n) = 2 \langle\langle D\, e \rangle\rangle(n)}$$

Y(n) is the mean exponential growrh factor of nearby orbits (MEGNO), Cincotta et al (2001)

In the next slides we compare the errors for the 2D Hénon map

LE normalized Lyapunov error $e_L(n)$. The error $e_L(n,\eta)$ is avoided since η introduces a bias.

FE and RE The exact formulae involve the tangent map

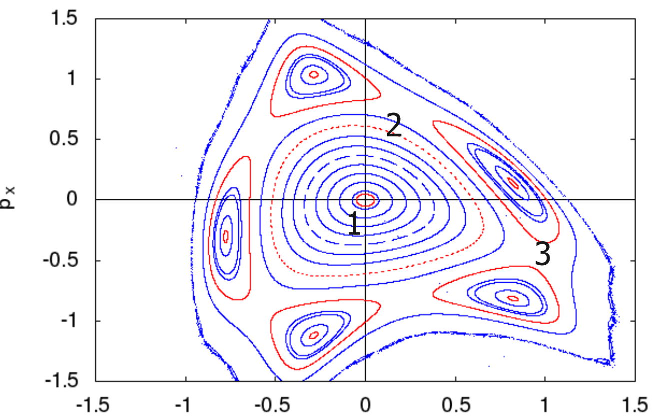
REM reversibility error (method) due to round off.

The computation of REM requires just n iterates of the map followed by n iterates of the inverse map. One can avoid the tangent map in computing LE choosing $\varepsilon=10^{-14}$ and two orthonormal vectors η

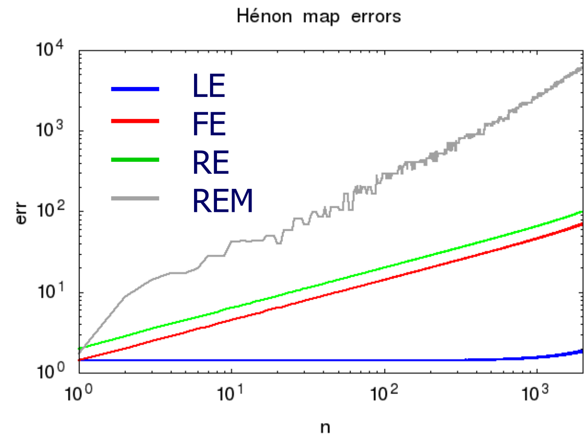
Birkhoff normal forms and betatronic motion

Power law growth of e(n) for Hénon map

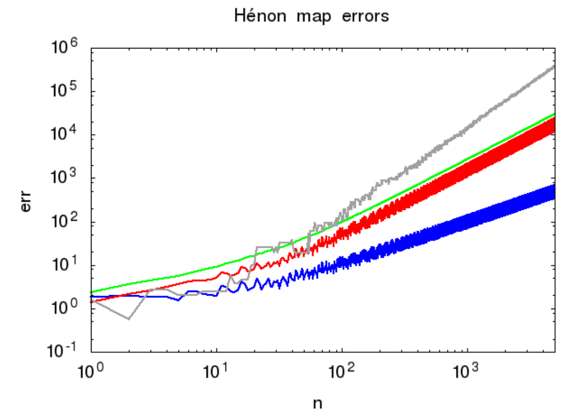
orbits $\nu = \sqrt{2} - 1$



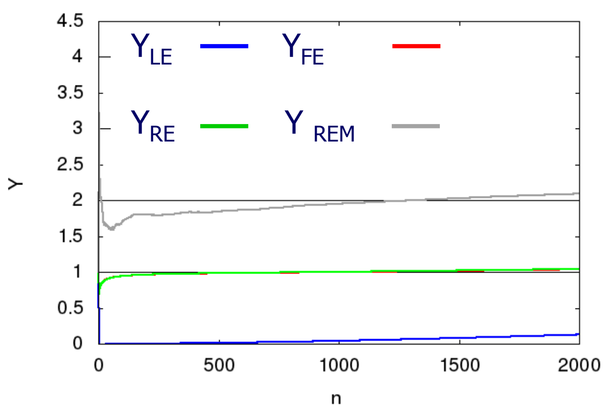
errors 1: $x_0=0.05$ $p_0=0$



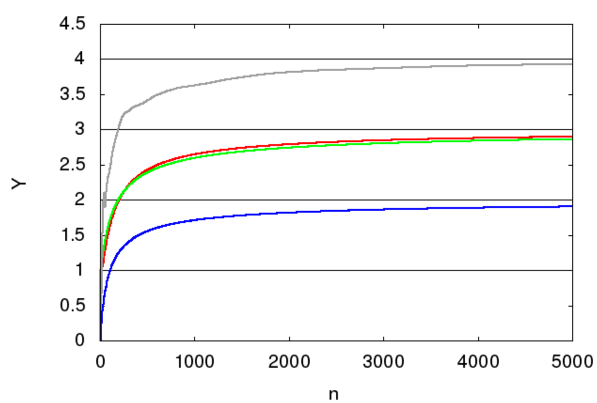
2: $x_0=0.6$ $p_0=0$



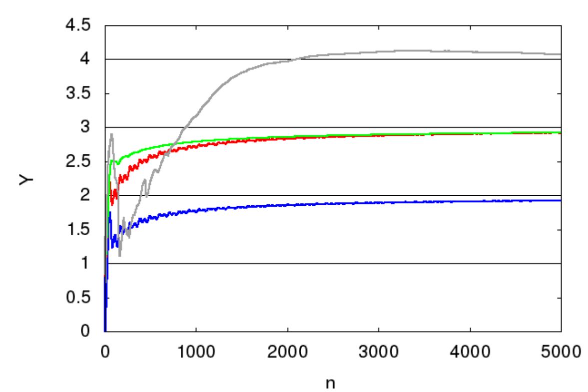
averages 1: $x_0=0.05$ $p_0=0$



2: $x_0=0.6$ $p_0=0$



3: $x_0=0.75$ $p_0=0$

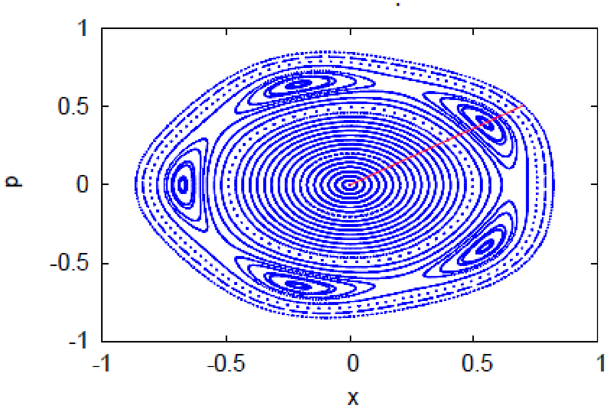


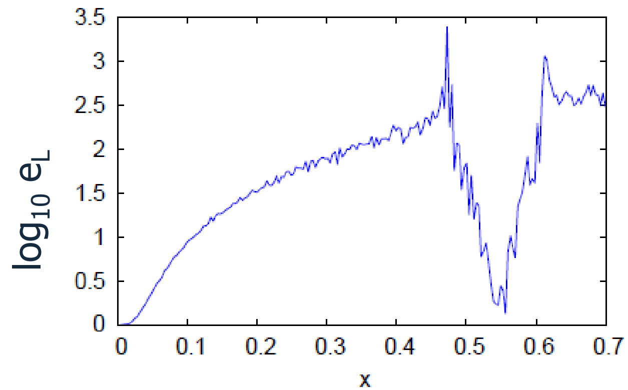
Birkhoff normal forms and betatronic motion

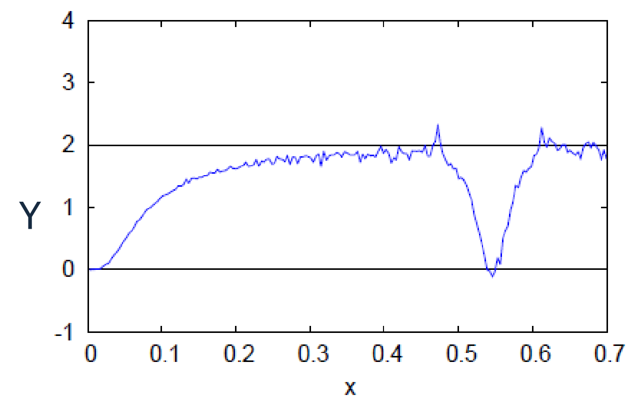
Integrable map. Near a stable resonance $\nu = m/q + \varepsilon/2\pi$ with $q > 4$ the resonant normal form gives a good approximation up to the dynamic aperture.

$$H(x,p) = \frac{\Omega_2}{2}\left(J + \frac{\epsilon}{\Omega_2}\right)^2 + J^{q/2}\left(A + B\,T_q\left(\frac{x}{\sqrt{2J}}\right)\right)$$

The errors growth follow a power law $e_L(n) \sim n\,|\Omega'(J)|\,J$. Near the separatrix $\Omega(J) \sim 1/\log(J_s - J)$ and $\Omega' \sim (J_s - J)^{-1}$ still a power law







Orbits of H near 1/5 resonance

Lyapunov error e_L (n)

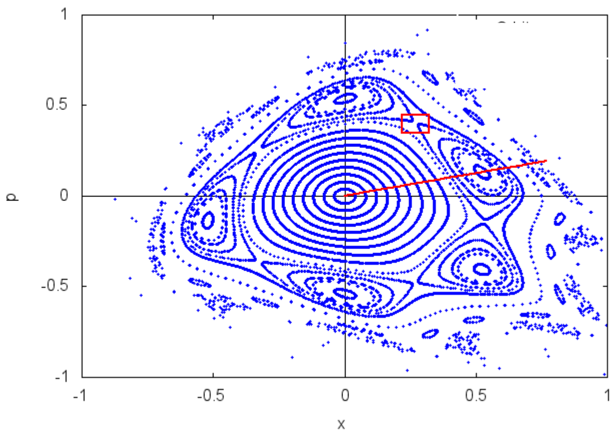
MEGNO average Y_L (n)

$\mathbf{x}_n = \exp(n\,\Delta t\,D_H)\,\mathbf{x}_0$ with $\Delta t = 0.2$ n=1000

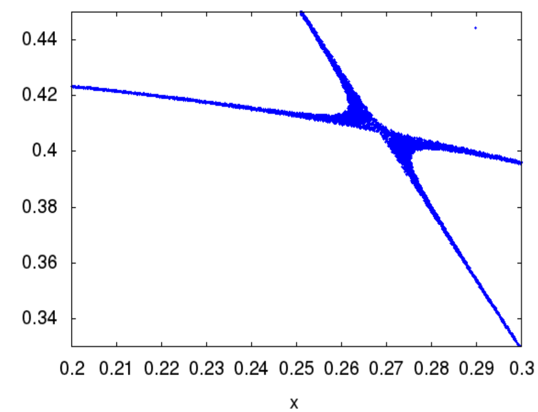
Birkhoff normal forms and betatronic motion

For the Hénon map the boundary of a chain of islands is a chaotic layer, where the errors growth is exponential. Plots for e(n), Y(n) when $\nu=0.21$

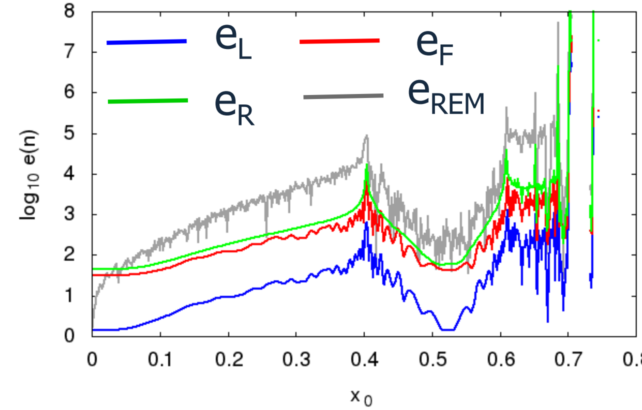
Obits



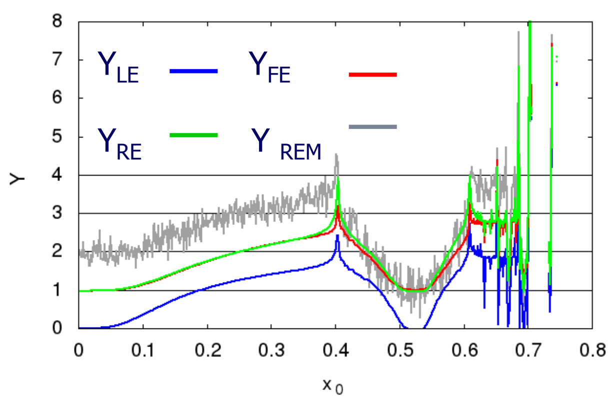
magnification



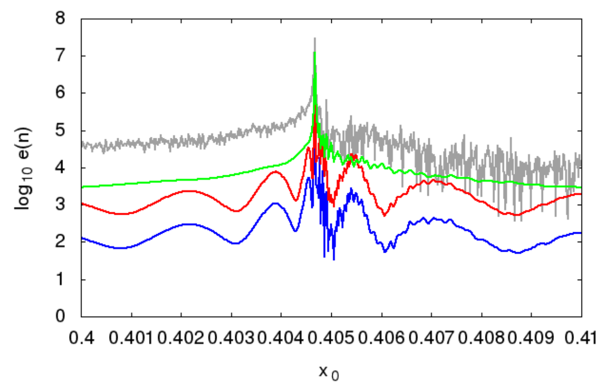
errors $\log_{10} e(n)$



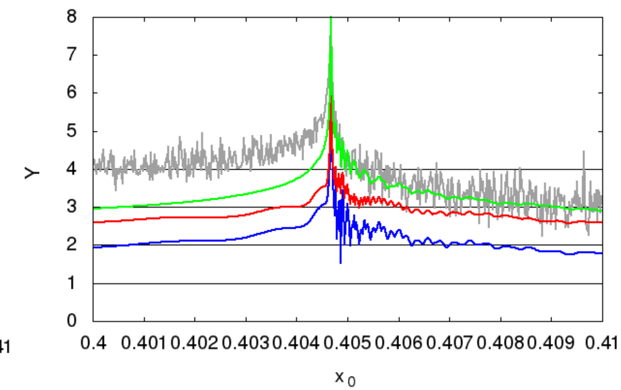
averages Y(n)



errors zoom



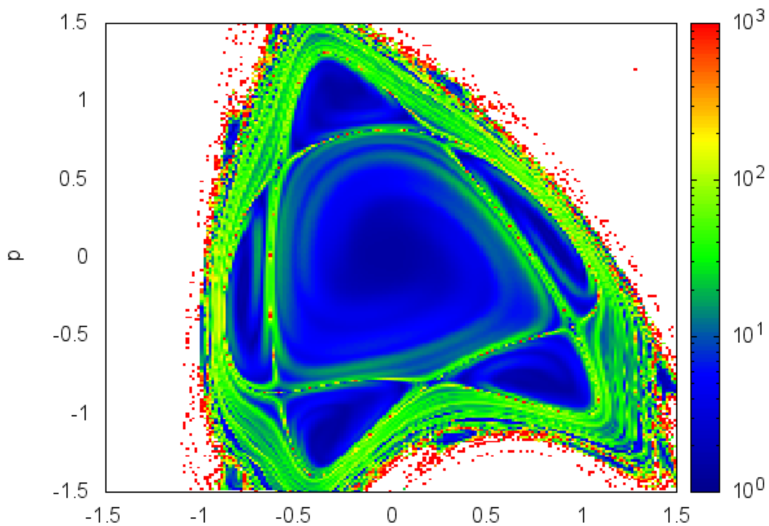
averages Y(n) zoom



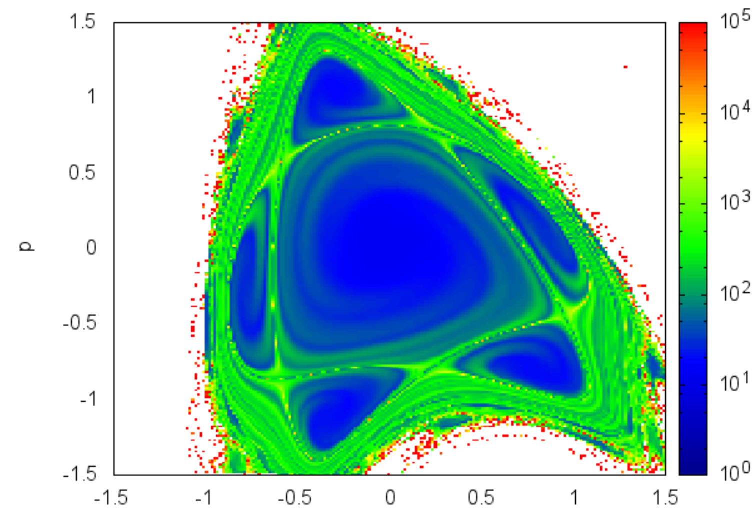
Birkhoff normal forms and betatronic motion

Normalized REM error color plots for $\nu = \sqrt{2} - 1$ $N=100$

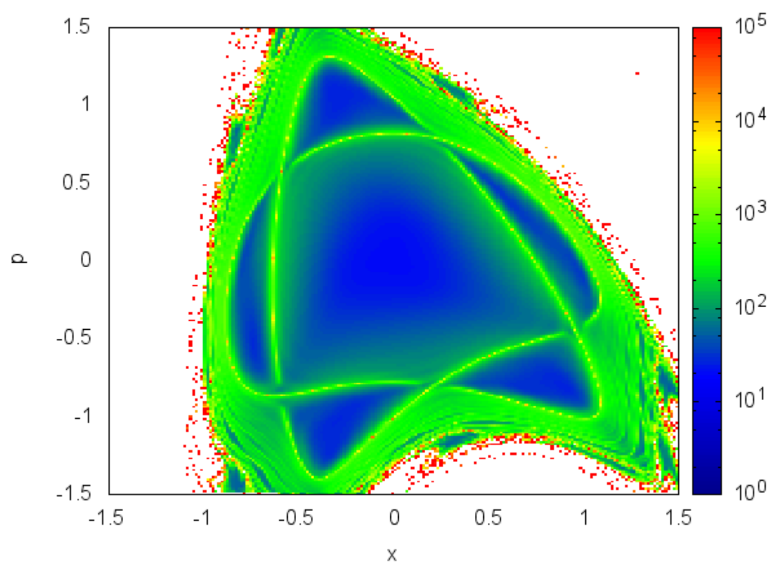
LE



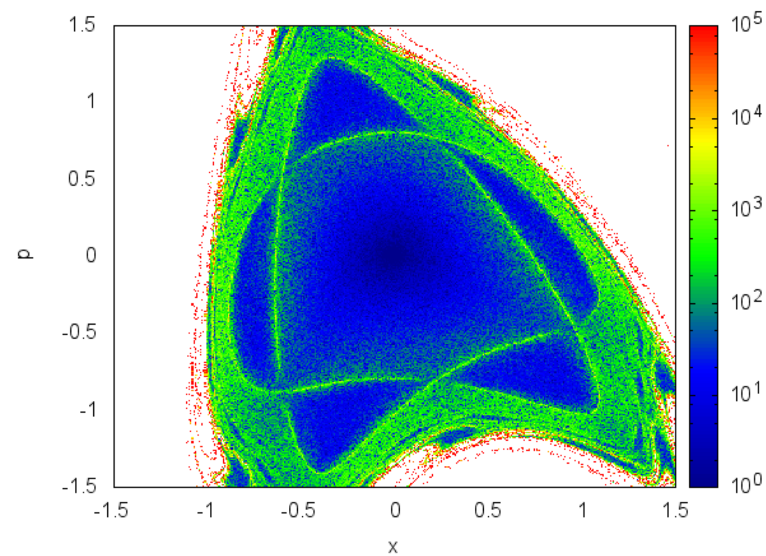
FE



RE

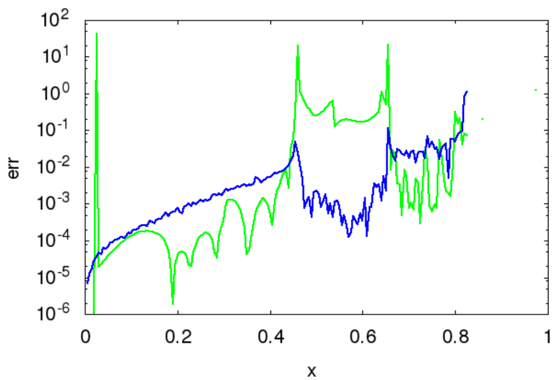


RE round off

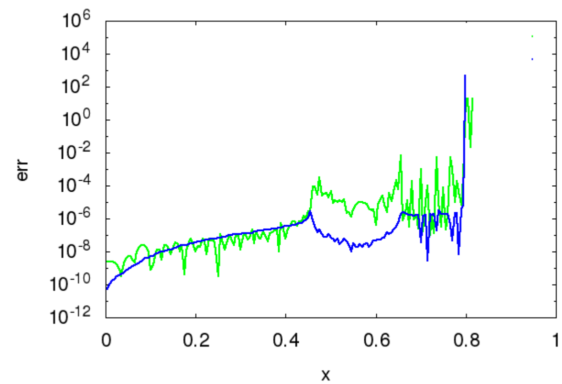


Comparison of frequency map error and REM

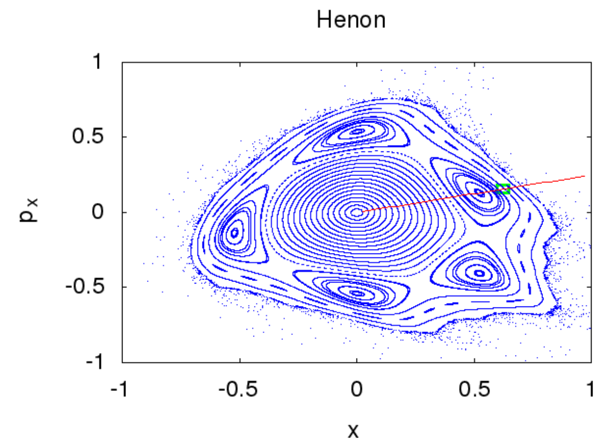
Frequency map error   REM

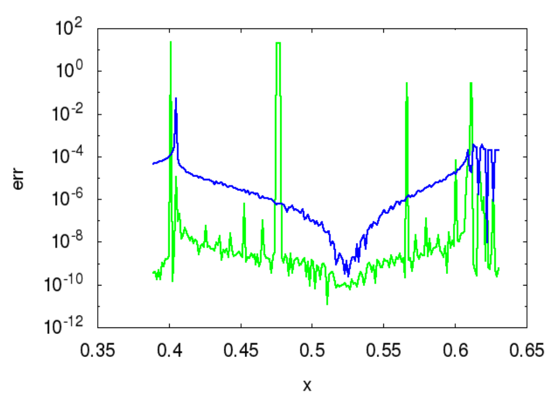


$n= 2^{11}$

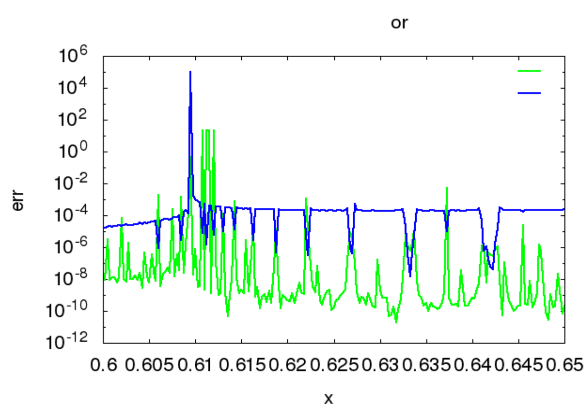


$n=2^{14}$

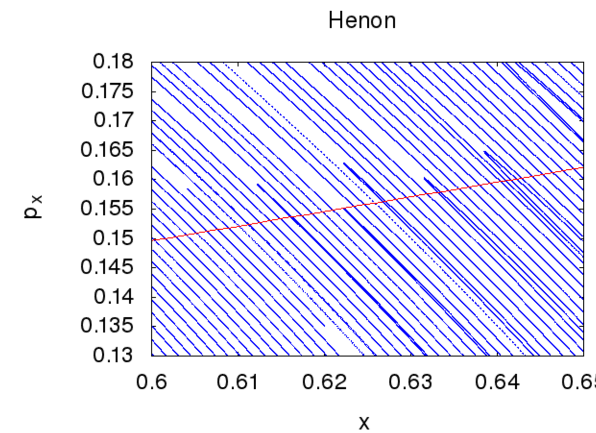




$n= 2^{14}$

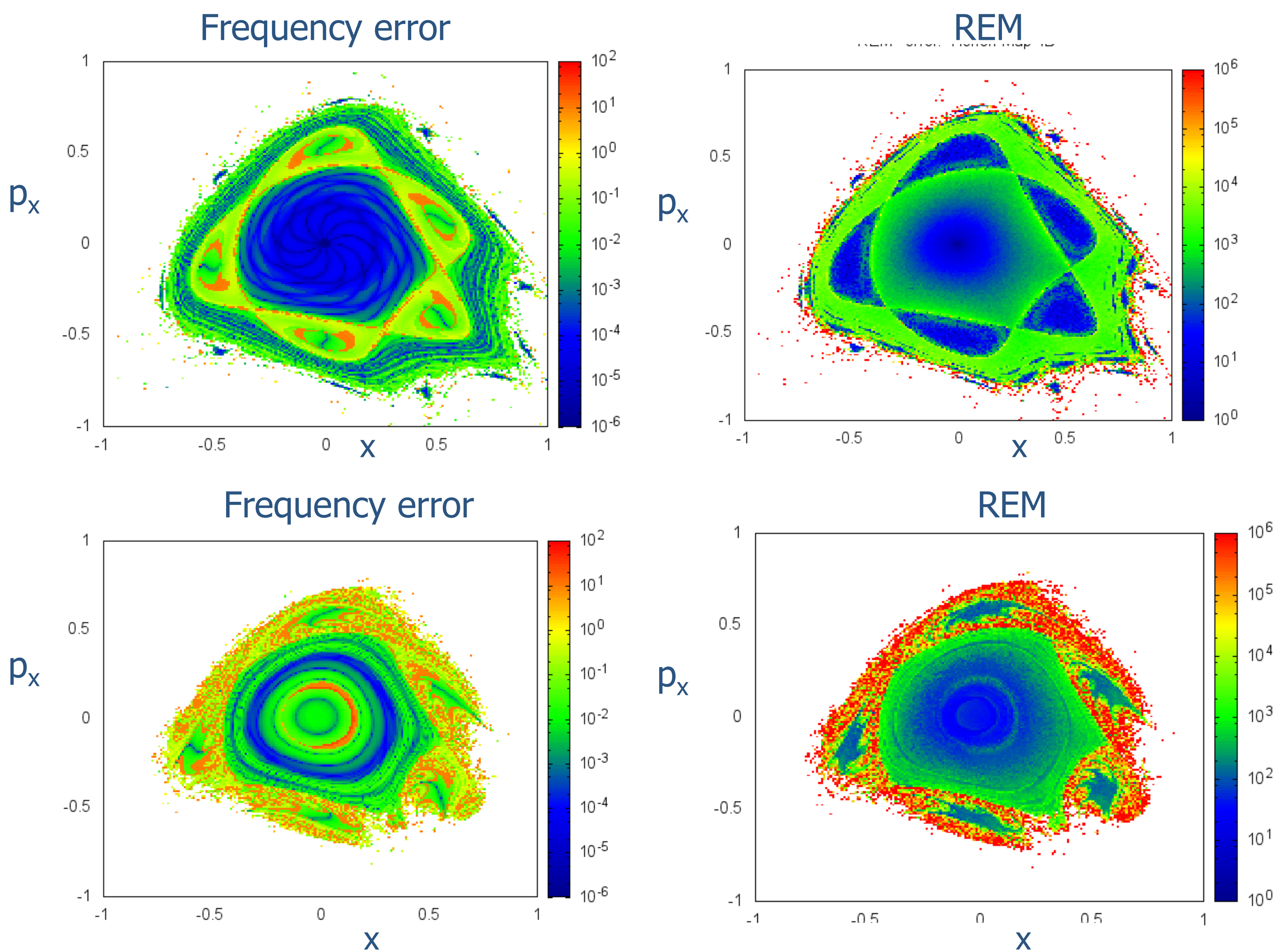


$n=2^{14}$



Frequency error and REM: 4D Hénon map tunes $\nu_1=0.21$ $\nu_2=0.25$

x, p_x plane



$y_0=p_{y0}=0$

$y_0=0.15$

$p_{y0}=0$

CONCLUSIONS

Birkhoff normal forms and betatronic motion

Beam and celestial dynamics share the same tools

- The Birkhoff normal forms describe the non linear betatron motion
- Resonance induced singularities make the series asymptotic
- Applications: tune diagrams, dynamic aperture, slow extraction

- The Lyapunov and noise induced errors provide stability portraits
- The round off reversibility error gives comparable results
- The error analysis applies in absence of a first integrals H

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